DATE OF EXAM **September** 14, 2016 **Solution**
SUBJECT NAME **Optimization** **MIDTERM Exam** **-** **Semester I**

1. *Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{pmatrix}$ *and* $b = \begin{pmatrix} 14 \\ 18 \\ 20 \end{pmatrix}$. *Obtain the LU decomposition of A and use it to solve the system* $Ax = b$.

   Solution: Let $A = LU$ with $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$. Then equating the entries of the matrix on both sides, by a simple calculation we find that

   $$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{pmatrix}.$$

   Now let $y = Ux$. Then we first solve for $Ly = b$. A simple calculation leads to $y = \begin{pmatrix} 14 \\ -10 \\ -72 \end{pmatrix}$.

   Then we solve for $Lx = y$ to get $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. □

2. *(i) Let a, b be column vectors in* $\mathbb{R}^n$ *and consider the matrix* $A = I + ab^t$. *Show that* $A^2 + \alpha A + \beta I = \bigcirc$

   Solution: Note that if $x \in \mathbb{R}^n$ is orthogonal to b, then $Ax = x + ab^t x = x + < b, x > a = x$. Since the orthogonal complement of b is of dimension $n-1$, it follows that the characteristic polynomial of A is of the form $(\lambda - 1)^{n-1}(\lambda - r)$ for some real number r. Then it follows that A satisfies some second degree equation of the form $A^2 + \alpha A + \beta I = \bigcirc$. Therefore, when A^{-1} exists, $A^{-1} = -\frac{1}{\beta}(A + \alpha I)$.

   (ii) Consider the $n \times n$ *matrix*

   $$B = \begin{pmatrix} a & b & b & ... & b \\ b & a & b & ... & b \\ b & b & a & ... & b \\ ... & ... & ... & ... & ... \\ b & b & b & ... & a \end{pmatrix},$$

   where a, b are real numbers such that $a \neq b$ *and* $a + (n-1)b \neq 0$. *Solve the system* $Bx = c$ *for a given vector* $c \in \mathbb{R}^n$.

   Solution: Note that $B = (a - b)B_1$, where $B_1 = \left[I + \frac{b}{a-b} \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & ... & 1 \end{pmatrix} \right]$.

   Then as in (i), $B_1^2 + \alpha B_1 + \beta I = \bigcirc$ with $\alpha = -(1 + a + (n-1)b)$ and $\beta = (a + (n-1)b)^2$. It is easy to see that B_1 is invertible and therefore $B_1^{-1} = -\frac{1}{\beta}(B_1 + \alpha I)$. Then $\frac{1}{a-b} B_1^{-1} c$ solves the equation $Bx = c$.

3. *(a) Let* $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. *Find a* 4×3 *matrix* Q *satisfying* $Q^t Q = I_3$ *and an upper triangular* 3×3 *matrix* R *with all diagonal elements positive such that* $A = QR$.

Solution: Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Let $e_1 = \frac{v_1}{||v_1||} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, $\tilde{e_2} = v_2 - < v_2, e_1 > e_1$ and $e_2 = \frac{\tilde{e_2}}{||\tilde{e_2}||} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$. $\tilde{e_3} = v_3 - < v_3, e_1 > e_1 - < v_3, e_2 > e_2$ and $e_3 = \frac{\tilde{e_3}}{||\tilde{e_3}||} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$. Then if $Q = [e_1\ e_2\ e_3] = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$, we have $Q^t Q = I_3$, since $\{e_1, e_2, e_3\}$ is an orthonormal set of vectors. Now let $R = \begin{pmatrix} < v_1, e_1 > & < v_2, e_1 > & < v_3, e_1 > \\ 0 & < v_2, e_2 > & < v_3, e_2 > \\ 0 & 0 & < v_3, e_3 > \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$. Then it is clear from the construction of e_1, e_2 and e_3 that $A = QR$, where Q and R are as described above.

(b) Suppose V *is an inner product space over* $\mathbb{C}$ *and* P *is a projection in* V. *If* $< Px, x > \leq ||x||^2$ *for all vectors* $x \in V$, *show that* P *is an orthogonal projection.*

Solution: Let $x \in \text{Im}P$ and $y \in \text{Ker}P$. Then foe any $\lambda \in \mathbb{C}$, we have

$$< P(x + \lambda y), x + \lambda y > \leq ||x + \lambda y||^2.$$

Since $x \in \text{Im}P$ and P is a projection we have $Px = x$. On the other hand since $y \in \text{Ker}P$, we have $Py = 0$. Hence from the above inequality it follows that

$$||x||^2 + \bar{\lambda} < x, y > \leq ||x||^2 + |\lambda|^2 ||y||^2 + \lambda < y, x > + \bar{\lambda} < x, y > .$$

So we see that $\lambda < y, x >$ has to be real. But since $\lambda \in \mathbb{C}$ is arbitrary, it follows that $< y, x >= 0$. Hence x is orthogonal to y. Since $x \in \text{Im}P$ and $y \in \text{Ker}P$ are arbitrary, it follows that P is an orthogonal projection.

4. *Let* u_1, u_2, v_1, v_2 *be non-zero column vectors in* $\mathbb{R}^n$ *and define* $P = u_1 u_2^t + v_1 v_2^t$. *Derive sufficient conditions on the given vectors so that* P *is a projection. Further, derive sufficient conditions on the given vectors so that* P *is an orthogonal projection.*

Solution: Note that given any vector x, $Px =< u_2, x > u_1 + < v_2, x > v_1$. So $\text{Im}P$ is the subspace spanned by the vectors u_1 and v_1. So if $Pu_1 = u_1$ and $Pv_1 = v_1$, then P is a projection. Now

$$Pu_1 =< u_2, u_1 > u_1 + < v_2, u_1 > v_1$$

and

$$Pv_1 =< u_2, v_1 > u_1 + < v - 2, v_1 > v_1.$$

Hence the sufficient conditions for P to be a projection are u_2, v_2 are in the orthogonal complement of the subspace spanned by the vectors u_1 and v_1 and $< u_2, u_1 >= 1$, $< v_2, v_1 >= 1$.

To do the second part we know that P is an orthogonal projection if and only if P^t is the same linear operator. Now $P^t = u_2 u_1^t + v_2 v_1^t$. Again writing the equations as above, a simple calculation shows that P^t is the same projection as P if $u_1 = u_2$, $v_1 = v_2$ and $\{u_1, v_1\}$ is an orthonormal set of vectors.

5. *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ *and* $b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \in \mathbb{R}^3$. *Let* $c = min\, ||Ax - b||$, *where the minimum is taken over all the column vectors* x *in* $\mathbb{R}^3$. *Determine* c *and obtain the solution* x *with least norm such that* $c = ||Ax - b||$.

Solution: $A = BC$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Let $B^+ = (B^t B)^{-1} B^t = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$ and $C^+ = C^t (CC^t)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$. Let $A^+ = C^+ B^+ = \frac{1}{9}\begin{pmatrix} 5 & -4 & 1 \\ -4 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.
Then we know that $AA^+A = A$ and A^+A is an orthogonal projection. $Ax - b = AA^+(Ax - b) + (I - AA^+)(Ax - b)$ and therefore,

$$||Ax - b||^2 = ||AA^+(Ax - b)||^2 + ||(I - AA^+)(Ax - b)||^2 = ||Ax - AA^+b||^2 + ||b - AA^+b||^2.$$

Therefore, $||Ax - b|| \geq ||A(A^+b) - b||$ with equality holds when $x = A^+b$. Hence

$$c = ||A(A^+b) - b)|| = \left| \frac{1}{9}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & -4 & 1 \\ -4 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right|$$

$$= \left| \frac{1}{9}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 9 \\ 11 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right| = \frac{1}{9}\sqrt{29}.$$

Now let x be any solution of $Ax = c$. Then $x = A^+Ax + (x - A^+Ax)$. Which implies that

$$||x||^2 = ||A^+Ax||^2 + ||x - A^+Ax||^2, \text{ since } A^+A \text{ is an orthogonal projection}$$

$$= ||A^+AA^+b||^2 + ||x - A^+AA^+b||^2 = ||A^+b||^2 + ||x - A^+b||^2.$$

Hence $||x|| \geq ||A^+b||$, showing that $A^+b = \frac{1}{9}\begin{pmatrix} 1 \\ 9 \\ 11 \end{pmatrix}$ is the solution with least norm.

6. *Let* $A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$. *Explain in detail why the limit* $\lim_{k\to\infty} A^k$ *exists and find the limit.*

Solution: We note that A is non-negative and A^2 is strictly positive, hence A is irreducible. By Calculating the characteristic polynomial of A we see that 1 is the dominant eigenvalue of A and A^t and the other eigenvalues are real and of modulas less than 1. Also $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1/3 \\ 2/9 \\ 4/9 \end{pmatrix}$ are the eigenvectors corresponding to the eigenvalue 1 for A and A^t respectively and $< u, v >= 1$. It follows from $Au = u$, $A^t v = v$ and $< u, v >= 1$ that $A^k(I - uv^t) = A^k - uv^t$ and $(I - uv^t)A^k = A^k - uv^t$. Let $B^k = A^k - uv^t$. Note that $Bu = 0$ and if $Bx = \lambda x$, $A(I - uv^t)x = \lambda x$. So $A(I - uv^t)^2 x = \lambda(I - uv^t)x$, which implies that $A(I - uv^t)x = \lambda(I - uv^t)x$. Hence $\sigma(B) \subset \sigma(A) \cup \{0\}$, where $\sigma(B)$ denotes the spectrum of B. Next we show that $1 \notin \sigma(B)$. Suppose there exists x such that $Bx = x$, $x \neq 0$, then $A(I - uv^t)x = (I - uv^t)x$, which implies that $(I - uv^t)x = cu$ for some constant c. Therefore, $x = c'u$ for some c', which is a contradiction to the fact that $Bu = 0$. Hence $1 \notin \sigma(B)$. Hence the spectral radius of B is less than 1, so $\lim_{k\to\infty} B^k = 0$. Then it follows that $\lim_{k\to\infty} A^k = uv^t = \begin{pmatrix} 1/3 & 2/9 & 4/9 \\ 1/3 & 2/9 & 4/9 \\ 1/3 & 2/9 & 4/9 \end{pmatrix}$